Study of Quasiperiodic Solutions to Nonlinear Oscillations

Yoshitane SHINOHARA*, and Yoshiteru KAGENISHI*

ABSTRACT

We are concerned with the existence and uniqueness of the quasiperiodic solution to Duffing type equation. Our theorems make us to verify the existence and uniqueness of an exact quasiperiodic solution and know the error bound of approximation to the exact quasiperiodic solution.

The paper is concerned with the existence and uniqueness of the quasiperiodic solutions to nonlinear oscillations such as

$$\frac{d^2x}{dt^2}+2\mu\frac{dx}{dt}+\nu^2x=\varepsilon x^3+\sum_{k=1}^{m}(a_k\cos\nu_k t+b_k\sin\nu_k t), \tag{1}$$

where $\mu,\nu,\nu_k=2\pi/\omega_k$ and $\omega_k\,(k=1,2,\dots,m)$ are all positive, and $1/\omega_k$ are rationally linearly independent.

Putting $y=dx/dt$,

$$z=\begin{pmatrix}x\\ y\end{pmatrix},\quad A=\begin{pmatrix}0 & 1\\ -\nu^2 & -2\mu\end{pmatrix},\quad \phi(t)=\begin{pmatrix}0\\ \sum_{k=1}^{m}(a_k\cos\nu_k t+b_k\sin\nu_k t)\end{pmatrix},$$
$$\eta(z)=\begin{pmatrix}0\\ x^3\end{pmatrix},$$

the equation (1) can be written in the vector form as follows:

$$\frac{dz}{dt}=Az+\phi(t)+\varepsilon\eta(z). \tag{2}$$

The linear differential equation

$$Lz=\phi(t),\quad L=\frac{d}{dt}-A \tag{3}$$

satisfies the generalized exponential dichotomy for $\mu\neq 0$. Let $\Phi(t)$ be the fundamental matrix of the equation $Lz=0$ such that $\Phi(0)=E$ (unit matrix), we have

$$\|\Phi(t)\|\leq K_0e^{-\sigma_0 t}. \tag{4}$$

In the paper we use the ℓ_∞-norm $\|\cdot\|$ in Euclidean space and denote that $\|f\|=\sup_{t\in R}\|f(t)\|$ for any bounded function $f=f(t)$ on the real line R.

Then there exist a projection matrix $P(P=E)$, positive numbers σ_1,σ_2 and non-negative functions $C_1(t,s),C_2(t,s)$ such that

$$(5)\quad\begin{cases}
\text{(i)} & P^2=P, & \\
\text{(ii)} & \|\Phi(t)P\Phi^{-1}(s)\|\leq C_1(t,s)e^{-\sigma_1(t-s)} & \text{for}\quad t\geq s,\\
\text{(iii)} & \|\Phi(t)(E-P)\Phi^{-1}(s)\|\leq C_2(t,s)e^{-\sigma_2(s-t)} & \text{for}\quad t<s,\\
\text{(iv)} & \int_{-\infty}^{t}C_1(t,s)e^{-\sigma_1(t-s)}ds+\int_{t}^{+\infty}C_2(t,s)e^{-\sigma_2(s-t)}ds & \leq M.
\end{cases}$$

where

$$M=\begin{cases}\dfrac{K_0}{\mu-\sqrt{\mu^2-\nu^2}} & \text{if} \quad \mu>\nu>0,\\ \displaystyle\int_{-\infty}^{t} K_0(t-s)e^{-\mu(t-s)}ds & \text{if} \quad \mu=\nu,\\ \dfrac{K_0}{\mu} & \text{if} \quad 0<\mu<\nu.\end{cases}$$

Theorem 1 *Consider a nonlinear differential equation*

$$\frac{dz}{dt}=X(t,z), \tag{6}$$

where z and $X(t,z)$ are vectors and $X(t,z)$ is quasiperiodic in t with periods $\omega_1,\omega_2,\dots,\omega_m$ and is continuously differentiable with respect to z belonging to a region D of z-space.

Suppose that there is a continuously differentiable quasiperiodic function $z_0(t)$ with periods $\omega_1,\omega_2,\dots,\omega_m$ such that

$$\begin{cases} z_0(t)\in D,\\ \left\| \dfrac{dz_0(t)}{dt}-X[t,z_0(t)]\right\|\le r,\end{cases}$$

for all $t\in R$.

Further suppose that there are a positive number δ, a non-negative number $\kappa<1$ and a quasiperiodic matrix $A(t)$ with periods $\omega_1,\omega_2,\dots,\omega_m$ such that

(i) *the equation (3) satisfies a generalized exponential dichotomy,*

(ii) $D_\delta=\{z;\| z-z_0(t)\|\le\delta$ *for some* $t\in R\}\subset D,$

(iii) $\| \Psi(t,z)-A(t)\|\le\dfrac{\kappa}{M}$ *whenever* $\| z-z_0(t)\|\le\delta,$

and

(iv) $\dfrac{Mr}{1-\kappa}\le\delta,$

where $\Psi(t,z)$ is the Jacobian matrix of $X(t,z)$ with respect to z.

Then the equation (6) possesses a solution $z=\hat{z}(t)$ quasiperiodic in t with periods $\omega_1,\omega_2,\dots,\omega_m$ such that

$$\| z_0(t)-\hat{z}(t)\|\le\frac{Mr}{1-\kappa} \tag{7}$$

for all $t\in R$. Furthermore, to the equation (6) there is no other quasiperiodic solution belonging to D_δ besides $z=\hat{z}(t)$.

Let $z_0(t)$ be the quasiperiodic solution to the linear equation (3) and bounded by K as $\| z_0(t)\|\le K$.

Theorem 2 *If the parameter ε satisfies the inequality*

$$|\varepsilon|\le\frac{1}{13K^2M}, \tag{8}$$

the equation (1) possesses a quasiperiodic solution $z=\hat{z}(t)$ with periods $\omega_1,\omega_2,\dots,\omega_m$ such that

(9) $\| z_0(t) - \hat{z}(t) \| \leq K$

for all $t \in R$.

If the inequality (8) does not hold or the error estimation (9) is too crude, we should compute a more accurate approximation than $z_0(t)$. For this purpose the Galerkin method is useful.

Starting from the solution $z = z_0(t)$ of the equation (3), we can compute the following Galerkin approximation of n-th order to the solution $x(t)$ of the equation (1):

(10) $$x_n(t) = \alpha(0,0) + \sum_{r=1}^{n} \sum_{|p|=r} \{\alpha_p \cos(p,\nu)t + \beta_p \sin(p,\nu)t\},$$

$$(p,\nu) = \sum_{k=1}^{m} p_k \nu_k, \quad |p| = \sum_{k=1}^{m} |p_k|.$$

Our theorems above make us to verify the existence and uniqueness of an exact quasiperiodic solution and know the error bound of the approximation (6) to the quasiperiodic solution of Duffing's general type equation which appears in nonlinear oscillations.

References

[1] Y. Shinohara, Investigation of the quasiperiodic solutions to Duffing's general type equations, Bulletin of the Faculty of Engineering, The University of Tokushima, Vol. 44 (1999), 1-13.

(* Shikoku University, Japan)